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Axial conduction and the Graetz problem for a Bingham plastic in laminar tube flow

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Abstract—The problem of heat transfer for a Bingham plastic in laminar tube flow is studied with axial conduction both excluded and included. It is demonstrated that the assumption of ignoring axial conduction for Peclet numbers greater than 100 is erroneous, especially near the start of the heated zone of the pipe. A new technique, based on the Sturm–Liouville theory, is introduced to solve these problems. In contrast to other techniques previously published, it requires only simple eigenvalues and eigenfunctions and is easily generalized to include the effects of axial conduction—a difficult proposition for semi-analytic methods.

1. INTRODUCTION

A BINGHAM fluid is a substance which exhibits a yield stress τ_y that must be overcome before it will flow. Examples of such fluids are drilling mud, paint and grease. When these materials flow in a pipe, there may be a central region which moves as a solid (plug flow) but near the wall the usual parabolic velocity profile of a Newtonian fluid is observed.

This paper studies the heat transfer properties of the above fluids where axial conduction is first excluded and later included. The solution technique used is that proposed previously by Do [1] and Johnston and Do [2] based on the Sturm–Liouville integral transform theory and has the advantage over other solution methods suggested [3] that it requires only simple eigenvalues and eigenfunctions. Further, it is a simple matter to include the effects of axial conduction, the main purpose of this article.

Section 2 describes the dimensionless governing equation of heat transfer in a Bingham fluid with axial conduction included and Section 3 develops the solution where axial conduction is ignored. Section 4 reproduces the solutions of ref. [3] and looks at the number of terms required for a solution. The effects of axial conduction are included in the solution described in Section 5. Finally, the equations are solved for various Peclet numbers Pe and the results are considered in light of the assumption that axial conduction can be ignored for $Pe > 100$ [3].

2. GOVERNING EQUATIONS

The stress induced velocity gradient for a Bingham plastic in pipe flow is of the form [4]

$$\frac{\mathrm{d}U}{\mathrm{d}r} = \begin{cases} 0 & \text{for } \tau < \tau_y \\ -\dfrac{1}{\eta}(\tau-\tau_y) & \text{for } \tau > \tau_y \end{cases} \tag{1}$$

where U is the axial velocity component, r the radial coordinate, τ the local shear stress, τ_y the yield stress and η the Bingham viscosity. For constant properties, the dimensionless velocity profile can be expressed as

$$u(y) = \begin{cases} \dfrac{2(1-c)^2}{1-\dfrac{4}{3}+\dfrac{c^4}{3}} & (=u_1) \quad 0 \leqslant y \leqslant c \\[2ex] \dfrac{2(1-y^2-2c(1-y))}{1-\dfrac{4}{3}+\dfrac{c^4}{3}} & (=u_2) \quad c \leqslant y \leqslant 1 \end{cases} \tag{2}$$

where c is the ratio of the yield stress to wall shear stress (τ_y/τ_w). Note that $c = 1$ corresponds to plug flow $(U = U_{\max})$ and $c = 0$ corresponds to laminar flow.

If viscous dissipation is ignored, the steady flow constant property form of the energy equation is

$$\rho c_p U(r)\frac{\partial t}{\partial x} = k\left[\frac{1}{k}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right)+\frac{\partial^2 t}{\partial x^2}\right]. \tag{3}$$

The boundary conditions required to completely specify the problem are

$$\frac{\partial t}{\partial r}(x,0) = 0 \tag{4a}$$

$$t(x,R) = t_w \tag{4b}$$

$$t(0,r) = t_c \tag{4c}$$

$$t(\infty,r) = t_w \tag{4d}$$

where it has been assumed that the fluid eventually attains the wall temperature. These equations and boundary conditions can be expressed in dimensionless form as

$$\frac{u(y)}{2}\frac{\partial\theta}{\partial z} = \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial\theta}{\partial y}\right)+\frac{1}{Pe^2}\frac{\partial^2\theta}{\partial z^2} \tag{5}$$

NOMENCLATURE

c	ratio of yield stress to wall stress, τ_y/τ_w
c_p	specific heat at constant pressure
J_0, J_1	Bessel functions
k	thermal conductivity
$K_n(\xi_n, y)$	eigenfunction
N	number of terms required for convergence
Nu_x	local Nusselt number
Nu_m	average value of Nu_x between entrance and axial position x
Pe	Peclet number, $2U_{av}R/\alpha$
r	radial coordinate
R	radius of pipe
$t(x,r)$	temperature
t_b	bulk fluid temperature
t_e	entrance temperature
t_w	wall temperature
$U(r)$	axial velocity
U_{av}	average axial velocity
U_{max}	maximum axial velocity
u	U/U_{av}
x	axial coordinate
y	dimensionless radius, r/R
z	dimensionless axial coordinate, $(x/R)/Pe$.

Greek symbols

α	thermal diffusivity
η	Bingham viscosity
θ	dimensionless temperature, $(t_w - t(x,r))/(t_w - t_e)$
θ_b	dimensionless bulk fluid temperature, $(t_w - t_b)/(t_w - t_e)$
ξ_n	eigenvalue
ρ	fluid density
τ	local shear stress
τ_w	wall shear stress
τ_y	yield shear stress.

$$\frac{\partial \theta}{\partial y}(z, 0) = 0 \tag{6a}$$

$$\theta(z, 1) = 0 \tag{6b}$$

$$\theta(0, y) = 1 \tag{7a}$$

$$\theta(\infty, y) = 0. \tag{7b}$$

Equation (5) has been solved elsewhere [3, 5] where axial conduction was neglected. The approach used was a classical separation of variables technique leading to a Sturm–Liouville eigenvalue problem. The drawback associated with this approach is that the eigenproblem must be solved numerically to obtain the eigenvalues and eigenfunctions. This paper, using the approach already successfully used by Do [1] and Johnston and Do [2], solves equation (5) by exploiting the eigenproblem which arises from the physical geometry of the situation (in this case a pipe). The resulting eigenvalues and eigenfunctions are in terms of Bessel functions. It will be demonstrated that this approach is easily implemented and that including finite Peclet numbers is a minor extension. The paper also shows that non-smooth velocity profiles can be handled by this technique.

3. SOLUTION METHOD

Consider the situation where axial conduction is ignored, that is large Pe. The governing equation becomes

$$\frac{u(y)}{2}\frac{\partial \theta}{\partial z} = \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial \theta}{\partial y}\right) \tag{8}$$

subject to radial boundary conditions (6) and axial initial condition (7a). From the second-order radial differential operator

$$\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y}\right)$$

and its associated homogeneous boundary conditions (6) define the eigenproblem

$$\frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y}\left(y\frac{\mathrm{d}K_n}{\mathrm{d}y}\right) + \xi_n^2 K_n = 0 \tag{9}$$

$$\frac{\mathrm{d}K_n}{\mathrm{d}y}(0) = 0 \tag{10a}$$

$$K_n(1) = 0. \tag{10b}$$

The solution for this eigenproblem is

$$K_n(y) = J_0(\xi_n y) \tag{11}$$

where the eigenvalues are the solutions of the transcendental equation

$$J_0(\xi_n) = 0. \tag{12}$$

Next, define a finite Sturm–Liouville integral transform for the kernel K_n

$$\langle \theta, K_n \rangle = \int_0^1 y\theta(z, y)K_n(y)\,\mathrm{d}y \tag{13}$$

and its inverse

$$\theta = \sum_{n=1}^{\infty} \frac{K_n(y)}{\langle K_n, K_n \rangle}\langle \theta, K_n \rangle. \tag{14}$$

Since the eigenproblem is based only on the radial differential operator, substituting this series representation of θ into the left-hand side of equation (8) results in

$$\frac{1}{2}\frac{\partial}{\partial z}\left[\sum_{n=1}^{\infty}\frac{u(y)K_n(y)}{\langle K_n,K_n\rangle}\langle\theta,K_n\rangle\right]=\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial\theta}{\partial y}\right). \quad (15)$$

Application of the integral transform (13) to this equation, gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left[\sum_{n=1}^{\infty}\frac{\langle u(y)K_n(y),K_m(y)\rangle}{\langle K_n,K_n\rangle}\langle\theta,K_n\rangle\right]=-\xi_m^2\langle\theta,K_m\rangle \quad (16)$$

which in turn can be expressed in matrix form by defining the following vectors and matrices:

$$\mathbf{\Theta}=\{\langle\theta,K_n\rangle\}$$

$$\mathbf{A}=\left\{\frac{1}{2}\frac{\langle u(y)K_n,K_m\rangle}{\langle K_n,K_n\rangle}\right\}$$

$$\mathbf{D}=\{\xi_m^2\delta_{nm}\}$$

as

$$\mathbf{A}\frac{\partial\mathbf{\Theta}}{\partial z}=-\mathbf{D}\mathbf{\Theta}. \quad (17)$$

Equation (17) is a system of first-order linear differential equations with initial conditions obtained by applying the integral transform (13) to equation (7a), i.e.

$$z=0,\quad \mathbf{\Theta}=\mathbf{\Theta}_0 \quad (18)$$

where $\mathbf{\Theta}_0=\{\langle 1,K_m\rangle\}$. The solution of equation (17) is

$$\mathbf{\Theta}=\mathbf{\Theta}_0\,\mathrm{e}^{-\mathbf{A}^{-1}\mathbf{D}z}. \quad (19)$$

Finally, given $\mathbf{\Theta}$, the inverse, θ, is evaluated from equation (14) or can be expressed in vector form as

$$\theta=\mathbf{f}\cdot\mathbf{\Theta} \quad (20)$$

where

$$\mathbf{f}=\left\{\frac{K_n(y)}{\langle K_n,K_n\rangle}\right\}. \quad (21)$$

Other quantities of interest, the bulk fluid temperature θ_b and local and average Nusselt numbers, Nu_x and Nu_m, respectively, are readily determined

$$\theta_b=2\int_0^1 yu(y)\theta\,\mathrm{d}y$$

$$=2\sum_{n=1}^{\infty}\frac{\langle\theta,K_n\rangle}{\langle K_n,K_n\rangle}\int_0^1 yu(y)K_n\,\mathrm{d}y \quad (22)$$

$$Nu_x=-\frac{2}{\theta_b}\frac{\partial\theta}{\partial y}(z,1)$$

$$=\frac{2}{\theta_b}\sum_{n=1}^{\infty}\frac{\langle\theta,K_n\rangle}{\langle K_n,K_n\rangle}\left.\frac{\mathrm{d}K_n}{\mathrm{d}y}\right|_{y=1} \quad (23)$$

$$Nu_m=\frac{1}{2z}\ln\left(\frac{1}{\theta_b}\right). \quad (24)$$

Two points should be emphasized at this stage. First consider the evaluation of $\langle u(y)K_n, K_m\rangle$. The definition of the integral transform implies that

$$\langle u(y)K_n,K_m\rangle=\int_0^1 u(y)J_0(\xi_n y)J_0(\xi_m y)\,\mathrm{d}y \quad (25)$$

where equation (11) has been employed. This integral must be evaluated in two parts due to the non-smooth velocity profile, that is

$$\langle u(y)K_n,K_m\rangle=\int_0^c u_1(y)J_0(\xi_n y)J_0(\xi_m y)\,\mathrm{d}y+\int_c^1 u_2(y)J_0(\xi_n y)J_0(\xi_m y)\,\mathrm{d}y. \quad (26)$$

This presents no major difficulties and a detailed evaluation of all integrals is included in the appendix.

The second point deals with the size of the system of linear equations (19). So far all matrices and vectors have been assumed infinite in dimension but for obvious practicalities this cannot be the case. Since the diagonal elements, a_{nn}, of $\mathbf{A}$ are greater than the off diagonal elements (a trend which is accentuated as n increases), equation (19) can be approximated by a finite system of size N (i.e. $\mathbf{A}$ is diagonally dominant for $n > N$). Hence these matrices and vectors can be expressed as

$$\mathbf{\Theta}=\{\langle\theta,K_1\rangle,\langle\theta,K_2\rangle,\ldots,\langle\theta,K_N\rangle\}^{\mathrm{T}}$$

$$\mathbf{A}=\left\{a_{nm}=\frac{1}{2}\frac{\langle u(y)K_n,K_m\rangle}{\langle K_n,K_n\rangle};\ n,m=1,2,\ldots,N\right\}$$

$$\mathbf{D}=\{\xi_m^2\delta_{nm};\ n,m=1,2,\ldots,N\}$$

$$\mathbf{\Theta}_0=\{\langle 1,K_n\rangle;\ n=1,2,\ldots,N\}^{\mathrm{T}}.$$

Equation (19) now gives the solution for the N-dimensional vector $\mathbf{\Theta}$.

To find $\langle\theta,K_n\rangle$ $(n=N+1,N+2,\ldots,\infty)$, the diagonal dominance of $\mathbf{A}$ is exploited. Hence equation (16) can be approximated by

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\langle u(y)K_n(y),K_n(y)\rangle}{\langle K_n,K_n\rangle}\langle\theta,K_n\rangle\right]=-\xi_n^2\langle\theta,K_n\rangle \quad (27)$$

for $n=N+1, N+2,\ldots,\infty$. The initial condition for equation (27) is

$$z=0;\ \langle\theta,K_n\rangle=\langle 1,K_n\rangle$$

and so, the solution of equation (27) is

$$\langle\theta,K_n\rangle=\langle 1,K_n\rangle\,\mathrm{e}^{-a_{nn}^{-1}\xi_n^2 z} \quad (28)$$

for $n=N+1, N+2,\ldots,\infty$.

For a given N, $\langle\theta,K_n\rangle$ for $n=1,2,\ldots,N$ are obtained from equation (19) and $\langle\theta,K_n\rangle$ for $n=N+1,N+2,\ldots$ are calculated from equation (28) until the infinite series (14) has converged. To choose the final value of N, equations (19) and (28) are solved in this manner with increasing N until two consecutive values of θ_b, Nu_x or Nu_m are arbitrarily close. Typically, N of the order of 30 gives accurate results for most values of z. A detailed discussion of this is given in the next section.

When $c = 1$, the second integral in equation (26) reduces to 0 and therefore

$$\langle u(y)K_n, K_m\rangle = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

giving the usual infinite series for plug flow. Hence, any value of N will yield an exact answer.

4. SOLUTION WITHOUT AXIAL CONDUCTION

To demonstrate this method it was decided to first repeat the calculations of Blackwell [3] and indicate the number of terms required to obtain a relative convergence of 0.0001%. The case of plug flow ($c = 1$) degenerates to a simple infinite series solution requiring no matrix manipulation.

Tables 1–6 give the values of Nu_x, Nu_m and θ_b for six values of c. The most interesting observation from these tables concerns the number of terms required for solution, N. For all values of c, as z increases N decreases, as would be expected, except that, for $z > 0.1$, N is the same for a given value of c. The behaviour of N as c increases changes with the value of z. For large values of z, the number of terms is almost constant but with a slight increase with increasing c. This trend is accentuated as z decreases to 0.0001. For $z = 0.0001$, the tendency is reversed.

5. AXIAL CONDUCTION

When axial conduction is included equation (5) is solved in full, subject to boundary conditions (6) and (7). A series solution of the form (4) is again assumed with the same kernel and eigenvalues. Substitution of this series into equation (5) and application of the integral transform (13) yields the second-order differential equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left[\sum_{n=1}^{\infty}\frac{\langle u(y)K_n(y), K_m(y)\rangle}{\langle K_n, K_n\rangle}\langle\theta, K_n\rangle\right] = -\xi_m^2\langle\theta, K_m\rangle + \frac{1}{Pe^2}\frac{\mathrm{d}^2\langle\theta, K_n\rangle}{\mathrm{d}z^2} \quad (29)$$

which can be expressed in terms of previously defined vectors and matrices as

$$\mathbf{A}\frac{\mathrm{d}\mathbf{\Theta}}{\mathrm{d}z} = \frac{1}{Pe^2}\frac{\mathrm{d}^2\mathbf{\Theta}}{\mathrm{d}z^2} - \mathbf{D}\mathbf{\Theta} \quad (30)$$

with boundary conditions

$$z = 0;\quad \mathbf{\Theta} = \mathbf{\Theta}_0 \quad (31a)$$

$$z \to \infty;\ \mathbf{\Theta} \to \mathbf{0}. \quad (31b)$$

Again $\mathbf{A}$ is diagonally dominant and so equation (30)

Table 2. Heat transfer results for flow of a Bingham plastic: $c = 0.8$

z	Nu_x	Nu_m	θ_b	N
0.0001	39.923	60.566	0.98797	119
0.0002	31.451	47.798	0.98107	94
0.0004	24.775	37.699	0.97030	80
0.0010	18.080	27.527	0.94644	65
0.0020	14.272	21.702	0.91686	55
0.0040	11.300	17.126	0.87196	45
0.0100	8.367	12.567	0.77776	35
0.0200	6.777	10.002	0.67026	33
0.0400	5.704	8.068	0.52442	24
0.1000	5.112	6.400	0.27802	23
0.2000	5.067	5.738	0.10074	23
0.4000	5.066	5.402	0.01328	23
1.0000	5.066	5.200	0.00003	23
2.0000	5.066	5.133	0.00000	23
4.0000	5.066	5.099	0.00000	23
10.0000	5.066	5.079	0.00000	10

Table 1. Heat transfer results for flow of a Bingham plastic: $c = 1.0$ (plug flow)

z	Nu_x	Nu_m	θ_b	N
0.0001	81.365	161.154	0.96828	—
0.0002	58.008	114.415	0.95527	—
0.0004	41.502	81.376	0.93697	—
0.0010	26.876	52.074	0.90109	—
0.0020	19.531	37.322	0.86132	—
0.0040	14.372	26.914	0.80629	—
0.0100	9.884	17.731	0.70144	—
0.0200	7.744	13.174	0.59040	—
0.0400	6.437	10.063	0.44708	—
0.1000	5.817	7.620	0.21785	—
0.2000	5.783	6.705	0.06843	—
0.4000	5.783	6.244	0.00677	—
1.0000	5.783	5.968	0.00001	—
2.0000	5.783	5.875	0.00000	—
4.0000	5.783	5.829	0.00000	—
10.0000	5.783	5.802	0.00000	—

Table 3. Heat transfer results for flow of a Bingham plastic: $c = 0.6$

z	Nu_x	Nu_m	θ_b	N
0.0001	33.313	50.483	0.98996	98
0.0002	26.262	39.863	0.98419	83
0.0004	20.704	31.464	0.97515	70
0.0010	15.134	22.996	0.95505	57
0.0020	11.973	18.151	0.92997	48
0.0040	9.521	14.351	0.89154	40
0.0100	7.150	10.590	0.80912	32
0.0200	5.892	8.502	0.71172	27
0.0400	5.038	6.942	0.57385	22
0.1000	4.539	5.592	0.32679	18
0.2000	4.494	5.048	0.13275	18
0.4000	4.494	4.771	0.02200	18
1.0000	4.494	4.604	0.00010	18
2.0000	4.494	4.549	0.00000	18
4.0000	4.494	4.521	0.00000	18
10.0000	4.494	4.504	0.00000	12

Table 4. Heat transfer results for flow of a Bingham plastic: $c = 0.4$

z	Nu_x	Nu_m	θ_b	N
0.0001	30.525	46.250	0.99080	96
0.0002	24.062	36.523	0.98550	78
0.0004	18.972	28.824	0.97721	65
0.0010	13.861	21.064	0.95875	53
0.0020	10.957	16.623	0.93567	45
0.0040	8.703	13.137	0.90024	38
0.0100	6.521	9.683	0.82394	30
0.0200	5.365	7.763	0.73306	26
0.0400	4.585	6.331	0.60261	22
0.1000	4.126	5.093	0.36107	18
0.2000	4.082	4.593	0.15929	18
0.4000	4.081	4.337	0.03114	15
1.0000	4.081	4.183	0.00023	15
2.0000	4.081	4.132	0.00000	15
4.0000	4.081	4.106	0.00000	15
10.0000	4.081	4.091	0.00000	15

Table 6. Heat transfer results for flow of a Bingham plastic: $c = 0.0$ (laminar flow)

z	Nu_x	Nu_m	θ_b	N
0.0001	28.256	42.813	0.99148	92
0.0002	22.273	33.811	0.98657	76
0.0004	17.556	26.681	0.97888	62
0.0010	12.825	19.499	0.96176	50
0.0020	10.131	15.382	0.94033	42
0.0040	8.037	12.150	0.90737	35
0.0100	6.002	8.942	0.83623	28
0.0200	4.916	7.155	0.75112	24
0.0400	4.173	5.812	0.62805	20
0.1000	3.710	4.640	0.39532	17
0.2000	3.658	4.156	0.18972	17
0.4000	3.657	3.906	0.04394	17
1.0000	3.657	3.757	0.00055	17
2.0000	3.657	3.707	0.00000	17
4.0000	3.657	3.682	0.00000	17
10.0000	3.657	3.667	0.00000	17

can be approximated by a finite $N \times N$ system of second-order differential equations which can be reduced to a first-order system (of size $2N \times 2N$) by defining

$$\frac{d\mathbf{\Theta}}{dz} = \mathbf{\Psi} \tag{32}$$

and

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{\Theta} \\ \cdots \\ \mathbf{\Psi} \end{bmatrix}. \tag{33}$$

The result is

$$\frac{d\mathbf{\Omega}}{dz} = \mathbf{E\Omega} \tag{34}$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{0} & \vdots & \mathbf{I} \\ \cdots & \cdots & \cdots \\ Pe^2\,\mathbf{D} & \vdots & Pe^2\,\mathbf{A} \end{bmatrix}. \tag{35}$$

Equation (34) is solved by obtaining the eigenvalues α_i and eigenvectors $\mathbf{v}_i$ of the matrix $\mathbf{E}$, giving a solution $\mathbf{\Omega}$ of the form

$$\mathbf{\Omega} = \sum_{i=1}^{N} \beta_i \mathbf{v}_i \, e^{\alpha_i z} \tag{36}$$

where the coefficients β_i are to be determined from boundary conditions (31). For positive eigenvalues, the corresponding β_i must be zero to satisfy boundary condition (31b). The remaining β_i are determined from boundary condition (31a) and so $\mathbf{\Omega}$ can be expressed as

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{\Theta} \\ \cdots \\ \mathbf{\Psi} \end{bmatrix} = \begin{bmatrix} \mathbf{S} \\ \cdots \\ \mathbf{T} \end{bmatrix} \mathbf{g} \tag{37}$$

where $\mathbf{g}$ is defined as

$$\mathbf{g} = \{e^{\alpha_i z}, \alpha_i < 0\}. \tag{38}$$

Hence the first N terms in the infinite series solution are given by

$$\mathbf{\Theta} = \mathbf{Sg}. \tag{39}$$

The remaining terms are obtained by solving the differential equation

$$\frac{a_{nn}}{2}\frac{d\langle\theta, K_n\rangle}{dz} = \frac{1}{Pe^2}\frac{d^2\langle\theta, K_n\rangle}{dz^2} - \xi_n^2\langle\theta, K_n\rangle \tag{40}$$

subject to the boundary conditions

$$z = 0;\ \langle\theta, K_n\rangle = \langle 1, K_n\rangle \tag{41a}$$

$$z \to \infty;\ \langle\theta, K_n\rangle \to 0. \tag{41b}$$

Table 5. Heat transfer results for flow of a Bingham plastic: $c = 0.2$

z	Nu_x	Nu_m	θ_b	N
0.0001	29.074	44.051	0.99123	94
0.0002	22.917	34.786	0.98619	76
0.0004	18.063	27.452	0.97828	63
0.0010	13.194	20.062	0.96067	51
0.0020	10.427	15.829	0.93865	43
0.0040	8.275	12.504	0.90481	36
0.0100	6.187	9.207	0.83182	29
0.0200	5.076	7.372	0.74463	24
0.0400	4.319	5.998	0.61887	20
0.1000	3.861	4.803	0.38269	18
0.2000	3.814	4.314	0.17808	17
0.4000	3.813	4.063	0.03875	17
1.0000	3.813	3.913	0.00040	17
2.0000	3.813	3.863	0.00000	17
4.0000	3.813	3.838	0.00000	17
10.0000	3.813	3.823	0.00000	17

Finally, this gives

$$\langle \theta, K_n \rangle = \langle 1, K_n \rangle \times \exp\left(\left[\frac{Pe^2\, a_{nn} - Pe\sqrt{(Pe^2\, a_{nn}^2 + 16\xi_n^2)}}{4}\right] z\right) \quad (42)$$

and so θ, θ_b, Nu_x and Nu_m can be determined as described previously.

6. SOLUTION WITH AXIAL CONDUCTION

The aim of this paper is to demonstrate the validity, or otherwise, of the assumption that axial conduction can be ignored for values of $Pe > 100$ [3].

The method of the previous section was applied for $Pe = 10$, 100 and 1000 over the z range 0.001 to 10 and for $c = 0$, 0.2, 0.4, 0.6 and 0.8. Figure 1 shows a typical plot of Nu_x vs z for $c = 0$. (Plots for other values of c are similar.) Clearly, as Pe increases the curves approach that of the solution for infinite Pe obtained in Section 4. With $Pe = 1000$, the curves are coincident. As z increases, curves for lower values of Pe converge to the curve for infinite Pe. The difference between the curves for $Pe = 100$ and infinite Pe at $z = 0.001$ is about 30%. This could lead to substantial errors.

A plot of Nu_m vs z for $c = 0.2$ (Fig. 2) again shows that $Pe = 1000$ is an adequate threshold for ignoring axial conduction. Comparing Nu_m curves for $Pe = 100$ and infinite Pe shows about a 20% difference at $z = 0.001$, but this approaches zero as z increases. Also, as $c \to 1$, the error decreases. However, for $Pe = 10$ the difference does not decrease to zero with increasing z, as with the Nu_x curves.

Figure 3 is a plot of θ_b vs z for $c = 0.4$ (this is representative of all values of c). Again, when $Pe = 1000$ the solution curve is identical to that obtained when axial conduction is ignored. There is a difference between the curves for $Pe = 100$ and infinite Pe at small values of z. The differences are accentuated when considering the curve for $Pe = 10$ and they exist for all but large values of z. Again the only safe choice for ignoring axial conduction is when $Pe > 1000$.

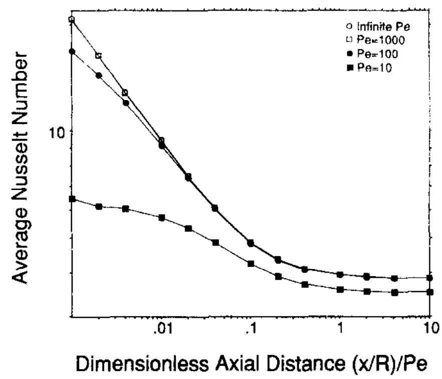

FIG. 2. Average Nusselt number vs dimensionless axial distance ($c = 0.2$).

These figures show that a valid threshold for ignoring axial conduction is $Pe = 1000$. Differences do exist for lower values of Pe at small values of z. In fact, potentially dangerous errors could result if axial conduction is ignored when Pe is of the order of 100, especially at small z.

The result of retaining Pe constant ($=10$) and allowing c to vary in a Nu_x vs z plot is shown in Fig. 4. For small values of z, Nu_x covers only a narrow range which spreads as z increases. As Pe increases the range of Nu_x values grows for small z [3].

Figure 5 is a plot of Nu_m vs z for $Pe = 100$. As Pe increases the ranges of values of Nu_m decreases at small values of z and spreads with decreasing Pe. Figure 6 is a plot of θ_b vs z for $Pe = 10$. This is typical for all values of Pe.

Finally, Tables 7–9 give the number of terms, N, required to obtain convergence as described previously for $Pe = 10$, 100 and 1000. As Pe decreases, the number of terms required to obtain convergence increases, especially for small values of z. At larger values of z, N is independent of Pe, and is always nearly independent of c.

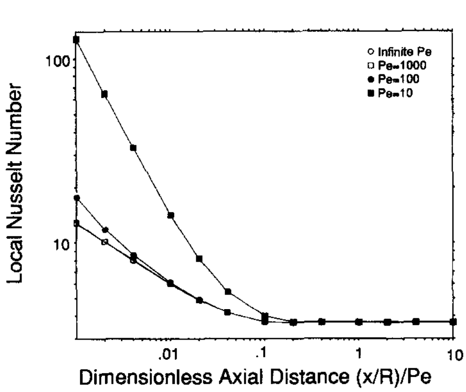

FIG. 1. Local Nusselt number vs dimensionless axial distance ($c = 0.0$).

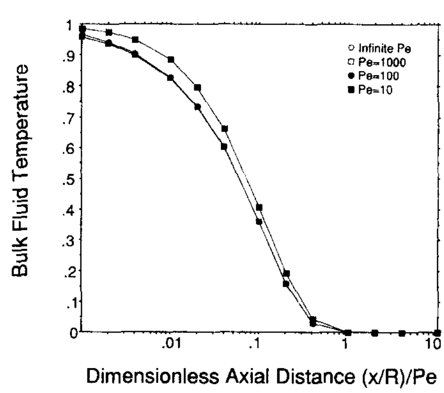

FIG. 3. Bulk fluid temperature vs dimensionless axial distance ($c = 0.4$).

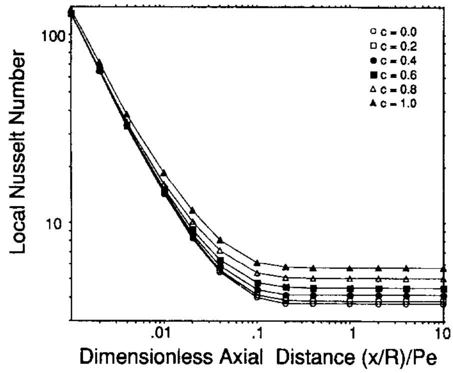

FIG. 4. Local Nusselt number vs dimensionless axial distance ($Pe = 10$).

Table 7. Number of terms required for convergence: $Pe = 10$

z	c 0.0	0.2	0.4	0.6	0.8
0.001	141	141	142	142	147
0.002	99	99	99	98	103
0.004	63	63	63	64	66
0.010	36	36	37	38	42
0.020	27	27	28	29	33
0.040	21	22	22	23	25
0.100	18	17	18	18	23
0.200	16	17	18	18	23
0.400	17	17	15	17	23
1.000	16	17	16	18	23
2.000	16	17	16	18	23
4.000	16	17	16	18	23
10.000	16	17	16	18	23

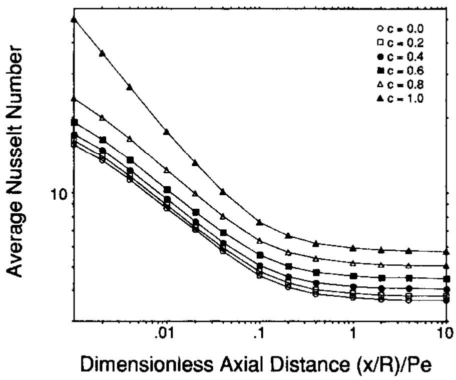

FIG. 5. Average Nusselt number vs dimensionless axial distance ($Pe = 100$).

Table 8. Number of terms required for convergence: $Pe = 100$

z	c 0.0	0.2	0.4	0.6	0.8
0.001	53	53	55	58	64
0.002	41	42	43	45	52
0.004	34	35	25	37	44
0.010	26	26	27	29	34
0.020	23	22	24	27	26
0.040	19	19	21	22	24
0.100	18	18	18	18	23
0.200	17	17	18	18	23
0.400	17	17	18	18	23
1.000	17	17	18	18	23
2.000	17	17	18	18	23
4.000	17	17	18	18	23
10.000	17	17	18	18	23

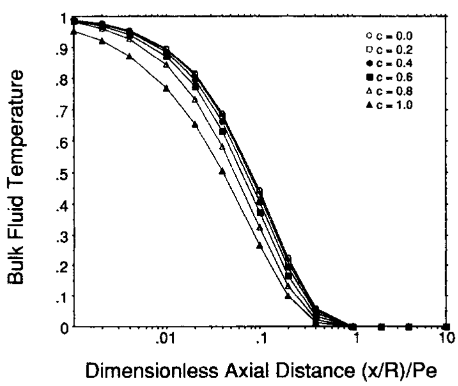

FIG. 6. Bulk fluid temperature vs dimensionless axial distance ($Pe = 10$).

Table 9. Number of terms required for convergence: $Pe = 1000$

z	c 0.0	0.2	0.4	0.6	0.8
0.001	45	45	47	50	57
0.002	37	38	40	42	48
0.004	31	32	33	36	43
0.010	25	26	27	29	34
0.020	23	22	25	27	32
0.040	20	20	22	22	24
0.100	17	18	18	18	23
0.200	16	17	18	18	23
0.400	16	17	18	18	23
1.000	16	17	18	18	23
2.000	16	17	18	18	23
4.000	16	17	18	18	23
10.000	16	17	18	18	23

7. CONCLUSION

This paper has presented a method based on Sturm–Liouville integral transforms to solve the equation for heat transfer in a Bingham fluid where axial conduction has been both excluded and included. The advantage of the method presented is that only simple eigenvalues and eigenfunctions are required and there is no need to solve the associated eigenproblem numerically as with other methods. The effectiveness of this technique for dealing with velocity profiles which are nonsmooth has also been demonstrated.

It has also been shown that in order to ignore axial conduction the Peclet number, Pe, must be greater than 1000. This is especially true for small values of z (in this case down to $z = 0.001$). Axial conduction can also be ignored when $Pe = 100$, but only when z is greater than 0.01. In fact, it could be postulated that axial conduction can be ignored when the axial positions of interest are greater than the inverse of the Peclet number. Ignoring this rule of thumb can lead to 20–30% differences in local and average Nusselt numbers over an order of magnitude change in axial position.

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APPENDIX. EVALUATION OF INTEGRALS

To evaluate $\langle uK_n, K_m \rangle$, integrals involving powers of y and combinations of Bessel functions must be evaluated [6]

$$\int_0^c yJ_0(\xi_n y)\,dy = \frac{cJ_1(\xi_n c)}{\xi_n} \tag{A1}$$

$$\int_c^1 yJ_0(\xi_n y)\,dy = \frac{1}{\xi_n}(J_1(\xi_n) - cJ_1(\xi_n c)) \tag{A2}$$

$$\int_0^c yJ_0(\xi_n y)J_0(\xi_m y)\,dy = \begin{cases} \dfrac{c}{\xi_n^2 - \xi_m^2}[\xi_n J_1(\xi_n c)J_0(\xi_m c) - \xi_m J_0(\xi_n c)J_1(\xi_m c)] \\ \qquad \xi_n \neq \xi_m \\ \dfrac{c^2}{2}[J_0^2(\xi_n c) + J_1^2(\xi_n c)] \\ \qquad \xi_n = \xi_m \end{cases} \tag{A3}$$

$$\int_c^1 yJ_0(\xi_n y)J_0(\xi_m y)\,dy = \begin{cases} \dfrac{c}{\xi_m^2 - \xi_n^2}[\xi_n J_1(\xi_n c)J_0(\xi_m c) - \xi_m J_0(\xi_n c)J_1(\xi_m c)] \\ \qquad \xi_n \neq \xi_m \\ \tfrac{1}{2}[J_1^2(\xi_n) - c^2 J_0^2(\xi_n c) - c^2 J_1(\xi_n c)] \\ \qquad \xi_n = \xi_m \end{cases} \tag{A4}$$

$$\int_c^1 y^3 J_0(\xi_n y)\,dy = \frac{1}{\xi_n^3}(\xi_n^2 J_1(\xi_n) - \xi_n^2 c^3 J_1(\xi_n c) - 4J_1(\xi_n) + 4J_1(\xi_n c) - 2\xi_n c^2 J_0(\xi_n c)) \tag{A5}$$

$$\int_c^1 y^3 J_0^2(\xi_n y)\,dy = -\frac{c^3}{\xi_n}J_1(\xi_n c)J_0(\xi_n c) - \frac{1}{\xi_n^2}(J_1^2(\xi_n) - c^2 J_1^2(\xi_n c)) + \frac{1}{6}\left(J_1^2(\xi_n) + \frac{4}{\xi_n^2}J_1^2(\xi_n)\right) - \frac{c^2}{6}\left(c^2 J_1^2(\xi_n c) + \frac{4}{\xi_n^2}J_1^2(\xi_n c) - \frac{4c}{\xi_n}J_1(\xi_n c)J_0(\xi_n c) + c^2 J_0^2(\xi_n c)\right) \tag{A6}$$

$$\int_c^1 y^2 J_0(\xi_n y)J_0(\xi_m y)\,dy = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\dfrac{\xi_n}{2}\right)^{2k} {}_2F_1(-k; -k; 1; \xi_m^2/\xi_n^2)}{k!(2k+3)} - c^3 \sum_{k=0}^{\infty} \frac{(-1)^k \left(\dfrac{\xi_n c}{2}\right)^{2k} {}_2F_1(-k; -k; 1; \xi_m^2/\xi_n^2)}{k!(2k+3)} \tag{A7}$$

$$\int_c^1 y^2 J_0^2(\xi_n y)\,dy = \tfrac{1}{3}\,{}_3F_4(\tfrac{1}{2}, 1, \tfrac{3}{2}; 1, 1, 1, \tfrac{5}{2}; -1) - \frac{c^3}{3}\,{}_3F_4(\tfrac{1}{2}, 1, \tfrac{3}{2}; 1, 1, 1, \tfrac{5}{2}; -c^2). \tag{A8}$$

CONDUCTION AXIALE ET PROBLEME DE GRAETZ POUR UN FLUIDE DE BINGHAM EN ECOULEMENT LAMINAIRE DANS UN TUBE

Résumé—Le problème du transfert thermique pour un fluide de Bingham en écoulement laminaire dans un tube est étudié avec conduction axiale soit négligée, soit prise en compte. On montre que l'hypothèse de conduction axiale négligée pour les nombres de Péclet supérieurs à 100 est erronée, particulièrement près du début de la zone chauffée du tube. Une technique nouvelle, basée sur la théorie de Sturm–Liouville, est introduite pour résoudre ces problèmes. Contrairement à d'autres techniques déjà publiées, elle demande simplement les valeurs propres et les fonctions propres et elle est facilement généralisable pour inclure les effets de la conduction axiale, ce qui est une affaire difficile pour beaucoup de méthodes semi-analytiques.

AXIALE WÄRMELEITUNG UND DAS GRAETZ-PROBLEM IN EINEM BINGHAM-MEDIUM BEI LAMINARER ROHRSTRÖMUNG

Zusammenfassung—Der Wärmeübergang in einem Bingham-Medium bei laminarer Rohrströmung wird mit und ohne Berücksichtigung der Längswärmeleitung untersucht. Es wird gezeigt, daß die Vernachlässigung der Längswärmeleitung für Peclet-Zahlen größer als 100 zu Fehlern führt—insbesondere zu Beginn der beheizten Zone im Rohr. Es wird ein neues Verfahren zur Lösung dieser Probleme auf der Grundlage der Sturm–Liouville-Theorie vorgestellt. Im Gegensatz zu anderen bereits veröffentlichten Verfahren werden nur einfache Eigenwerte und Eigenfunktionen benötigt. Es läßt sich leicht zur Berücksichtigung der Längswärmeleitung verallgemeinern—dies ist ein schwieriges Problem für einige halbanalytische Verfahren.

АКСИАЛЬНАЯ ТЕПЛОПРОВОДНОСТЬ И ЗАДАЧА ГРЕТЦА ДЛЯ ПЛАСТИЧЕСКОЙ СРЕДЫ БИНГАМА ПРИ ЛАМИНАРНОМ ТЕЧЕНИИ ПО ТРУБАМ

Аннотация—Исследуется задача теплопереноса для пластической трубы Бингама при ламинарном течении по трубам как с учетом, так и без учета кондуктивного аксиального теплопереноса. Показано, что пренебрежение аксиальной теплопроводностью в случае значений чисел Пекле, превышающих 100, является ошибочным, особенно у начала нагретого участка трубы. Для решения исследуемых задач используется новая методика на основе теории Штурма–Лиувилля. В отличие от ранее предложенных, данный метод требует только простых собственных значений и собственных функций и легко обобщается на случай учета эффектов аксиальной теплопроводности, что представляет трудность для некоторых полуаналитических методов.